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The Renormalization group in non-relativistic theories

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Abstract

The resummation of logarithms in Quantum Field Theories is a long tale plenty of successes, yet the resummation of logarithms in non-relativistic theories has remained elusive. This was the most frustrating, since the first quantum field theory log ever computed was the Lamb shift one. We briefly review recent progress on the resummation of logarithms of α, which appear in the physics of non-relativistic states, using effective field theories. We put special emphasis on the basic formalism.

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Heavy quark-antiquark systems near threshold are characterized by the small relative velocity v of the heavy quarks in their center of mass frame. This small parameter produces a hierarchy of widely separated scales: m (hard), mv (soft), mv^2 (ultrasoft), ... The factorization between them is efficiently achieved by using effective field theories, where one can organize the calculation as various perturbative expansions on the ratio of the different scales, effectively producing an expansion in v. The terms in these series get multiplied by parametrically large logs: $\ln v$, which can also be understood as the ratio of the different scales appearing in the physical system. Again, effective field theories are very efficient in the resummation of these large logs once a renormalization group (RG) analysis of them has been performed. Therefore, the use of effective field theories has opened the possibility to solve a problem, otherwise open, since the first non-relativistic logs appeared around fifty years ago (in the Lamb shift in the Hydrogen atom). We will review in this paper the recent progress achieved in potential NRQCD (pNRQCD) [1] on the issue of the resummation of $\ln v$ terms in the weak coupling regime (we will obviate any non-perturbative effects in what follows). In particular, we will focus on the theoretical aspects.

Potential NRQCD (pNRQCD) is defined by its particle content and cut-off $\nu_{pNR} = \{\nu_p, \nu_{us}\}$, where ν_p is the cut-off of the relative three-momentum of the heavy quarks and ν_{us} is the cut-off of the three-momentum of the gluons and light quarks. They satisfy the following inequalities: $|\mathbf{p}| \ll \nu_p \ll m$ and $\mathbf{p}^2/m \ll \nu_{us} \ll |\mathbf{p}|$, where typically $|\mathbf{p}| \sim mv$. Note that no gluons or light quarks with momentum of $O(|\mathbf{p}|)$ are kept dynamical in pNRQCD. The motivation to integrate out these degrees of freedom is that they do not appear as physical (on-shell) states near threshold. Nevertheless, they can appear off-shell and, since their momentum is of the order of the relative three-momentum of the heavy quarks, integrating them out produces non-local terms (potentials) in three-momentum space. Indeed, these potentials encode the non-analytical behavior in the transfer momentum of the heavy quark, $\mathbf{k} = \mathbf{p} - \mathbf{p}'$, of the order of the relative three-momentum of the heavy quarks.

In this paper, we will mainly consider the situation $v \sim \alpha_s$. It should be clear however that pNRQCD (in the weak coupling regime) is also valid in the situation $mv^2 \gg m\alpha_s^2$ as long as v is an small parameter. Nevertheless, the Coulomb resummation is not necessary in this case.

Formally, we can write the pNRQCD Lagrangian as an expansion in

$1/r(=1/r,\, p)$ and $1/m$ in the following way:

$$\mathcal{L}_{\rm pNRQCD} = \sum_{n=-1}^{\infty} r^n \tilde{V}_n^{(B)} O_n^{(B)} + \frac{1}{m} \sum_{n=-2}^{\infty} r^n \tilde{V}_n^{(B,1)} O_n^{(B,1)} + O\left(\frac{1}{m^2}\right), \qquad (1)$$

where the above operators and matching coefficients should be understood as bare. As for the renormalized quantities, we define V as the potentials and $\tilde{V}$ as the (almost) dimensionless constants in it. The latter are in charge of absorbing the divergences of the effective field theory. Therefore, they will depend on ν_p and ν_{us}. At next-to-leading order in the multipole expansion the Lagrangian can be written as[1]

$$\begin{aligned} L_{pNRQCD} = & \int d^3r d^3R tr\Big\{ S^\dagger \big\{ i\partial_0 - h_s \big\} S + O^\dagger \big\{ iD_0 - h_o \big\} O \qquad (2) \\ & + g V_A \mathbf{r}\cdot O \mathbf{E}(\mathbf{R},t) S^\dagger + g V_A \mathbf{r}\cdot O^\dagger \mathbf{E}(\mathbf{R},t) S \\ & + \frac{g}{2} V_B \mathbf{r}\cdot O O^\dagger \mathbf{E}(\mathbf{R},t) + \frac{g}{2} V_B \mathbf{r}\cdot O^\dagger O \mathbf{E}(\mathbf{R},t) \Big\}, \end{aligned}$$

where h_s and h_o are the singlet and octet quantum mechanical Hamiltonian. For illustration, at low orders (see Ref. [2] for notation and further details),

$$\begin{aligned} h_s \;=\; & c_k \frac{\mathbf{p}^2}{m} - c_4 \frac{\mathbf{p}^4}{4m^3} - C_f \frac{\alpha_{V_s}}{r} - \frac{C_f C_A D_s^{(1)}}{2mr^2} \\ & - \frac{C_f D_{1,s}^{(2)}}{2m^2} \left\{ \frac{1}{r}, \mathbf{p}^2 \right\} + \frac{C_f D_{2,s}^{(2)}}{2m^2} \frac{1}{r^3} \mathbf{L}^2 + \frac{\pi C_f D_{d,s}^{(2)}}{m^2} \delta^{(3)}(\mathbf{r}) \\ & + \frac{4\pi C_f D_{S^2,s}^{(2)}}{3m^2} \mathbf{S}^2 \delta^{(3)}(\mathbf{r}) + \frac{3 C_f D_{LS,s}^{(2)}}{2m^2} \frac{1}{r^3} \mathbf{L}\cdot\mathbf{S} + \frac{C_f D_{S_{12},s}^{(2)}}{4m^2} \frac{1}{r^3} S_{12}(\hat{\mathbf{r}}) \,, \qquad (3) \end{aligned}$$

in the equal mass case, where $C_f = (N_c^2-1)/(2N_c)$ and we will set $c_k = c_4 = 1$. A similar expression holds for the octet Hamiltonian changing the label of the matching coefficients ($s \to o$).

The matching process, which basically means the computation of the potentials, is carried out for a given external incoming (outcoming) momentum $\mathbf{p}$ ($\mathbf{p}'$). Therefore, one has to sum over all of them in the pNRQCD Lagrangian, since they are still physical degrees of freedom as far as their

[1] S and O stand for the heavy quarkonium bilinear field in a singlet and octet configuration under ultrasoft gauge transformations. They both depend on the relative, $\mathbf{r}$, and center of mass, $\mathbf{R}$, coordinate.

momentum is below ν_p. In position space, this means that an integral over $\mathbf{x}$, the relative distance between the heavy quarks, appears in the Lagrangian when written in terms of the heavy quark-antiquark bilinear field.

Within pNRQCD, integrals over $\mathbf{p}$ (or $\mathbf{x}$) appear when solving the Schrödinger equation that dictates the dynamics of the heavy quarkonium near threshold. At low orders, these integrals are finite effectively replacing $\mathbf{p}$ by $\sim m\alpha_s$. Nevertheless, at higher orders in quantum mechanics perturbation theory and/or if some singular enough operators are introduced (as it is the case of the heavy quarkonium production currents) singularities proportional to $\ln \nu_p$ appear. These must be absorbed by the potentials or by the matching coefficients of the currents.

Let us now describe the matching between QCD and pNRQCD within an RG framework. We first address the procedure that gives the running of the potentials. One first does the matching from QCD to NRQCD. The latter depends on some matching coefficients: $c(\nu_s)$ and $d(\nu_p, \nu_s)$, which can be obtained order by order in α_s (with $\nu_p = \nu_s$) following the procedure described in Ref. [3] for the $c(\nu_s)$ and [4] for the $d(\nu_p, \nu_s)$. ν_s is the ultraviolet cutoff of the three-momentum of the gluons in NRQCD. The $c(\nu_s)$ stand for the coefficients of the operators that already exist in the theory with only one heavy quark (ie. HQET) and the $d(\nu_p, \nu_s)$ stand for the coefficients of the four heavy fermion operators. The starting point of the renormalization group equation can be obtained from these calculations by setting $\nu_p = \nu_s = m$ (up to a constant of order one). In principle, we should now compute the running of ν_p and ν_s. The running of the $c(\nu_s)$ can be obtained using HQET techniques [5]. The running of the $d(\nu_p, \nu_s)$ is more complicated. At one-loop, ν_p does not appear and we effectively have $d(\nu_p, \nu_s) \simeq d(\nu_s)$, whose running can also be obtained using HQET-like techniques [2]. At higher orders, the dependence on ν_p appears and the running of the $d(\nu_p, \nu_s)$ becomes more complicated. Fortunately, we need not compute the running of d in this more general case because, as we will see, the relevant running of the d for near threshold observables can be obtained within pNRQCD.

The next step is the matching from NRQCD to pNRQCD. The latter depends on some matching coefficients (potentials). They typically have the following structure: $\tilde{V}(c(\nu_s), d(\nu_p, \nu_s), \nu_s, \nu_{us}, r)$. After matching, any dependence on ν_s disappears since the potentials have to be independent of ν_s. Therefore, they could be formally written as $\tilde{V}(c(1/r), d(\nu_p, 1/r), 1/r, \nu_{us}, r)$. These potentials can be obtained order by order in α_s following the procedure of Refs. [1, 6]. The integrals in the matching calculation would depend on

a factorization scale μ, which should correspond either to ν_s or to ν_{us}. In the explicit calculation, they could be distinguished by knowing the UV and infrared (IR) behavior of the diagrams: UV divergences are proportional to $\ln \nu_s$, which should be such as to cancel the ν_s scale dependence inherited from the NRQCD matching coefficients, and IR divergences to ν_{us}. In practice, however, as far as we only want to perform a matching calculation at some given scale $\mu = \nu_s = \nu_{us}$, it is not necessary to distinguish between ν_s and ν_{us} (or if working order by order in α_s without attempting any log resummation).

Before going into the rigorous procedure to obtain the RG equations of the potentials, let us first discuss their structure on physical grounds. The potential is independent of ν_s. This allows us to fix ν_s to $1/r$ that, in a way, could be understood as the matching scale for ${\nu_s}^2$. Therefore, $1/r$, the point where the multipole expansion starts, would also provide with the starting point of the renormalization group evolution of ν_{us} (up to a constant of order one). The running of ν_{us} can then be obtained following the procedure described in Refs. [7, 2]. Formally, the renormalization group equations of the matching coefficients due to the ν_{us}-dependence read

$$\nu_{us} \frac{d}{d\nu_{us}} \tilde{V} = B_{\tilde{V}}(\tilde{V}). \tag{4}$$

From a practical point of view one can organize the RG equations within an expansion in $1/m$. At $O(1/m^0)$, the analysis corresponds to the study of the static limit of pNRQCD, which has been carried out in Ref. [7]. Since $\tilde{V}_{-1} \neq 0$, there are relevant operators (super-renormalizable terms) in the Lagrangian and the US RG equations lose the triangular structure that we enjoyed for the RG equations of ν_s. Still, if $\tilde{V}_{-1} \ll 1$, a perturbative calculation of the renormalization group equations can be achieved as a double expansion in $\tilde{V}_{-1}$ and $\tilde{V}_0$, where the latter corresponds to the marginal operators (renormalizable interactions)). At short distances ($1/r \gg \Lambda_{\rm QCD}$), the static limit of pNRQCD lives in this situation. Specifically, we have $\tilde{V}_{-1} = \{\alpha_{V_s}, \alpha_{V_o}\}$, that fulfills $\tilde{V}_{-1} \sim \alpha_s(r) \ll 1$; $\tilde{V}_0 = \alpha_s(\nu_{us})$ and $\tilde{V}_1 = \{V_A, V_B\} \sim 1$. Therefore, we can calculate the anomalous dimensions order by order in $\alpha_s(\nu_{us})$. In addition, we also have an expansion in $\tilde{V}_{-1}$. Moreover, the specific form of the pNRQCD Lagrangian severely constrains the RG equations general

[2] In practice, the potential is often first obtained in momentum space so that one could then set $\nu_s = k$. Note, however, that this is not equivalent to fix $\nu_s = 1/r$, since finite pieces will appear after performing the Fourier transform.

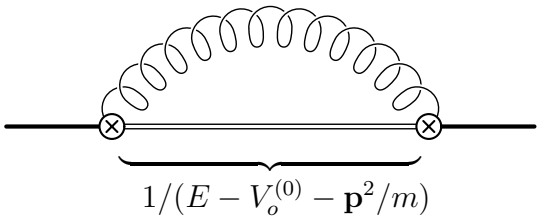


Figure 1: The UV divergences of this diagram in pNRQCD give the leading non-trivial ultrasoft running of α_{V_s}, $D_s^{(1)}$, $D_s^{(2)}$.

structure. Therefore, for instance, the leading non-trivial RG equation of α_{V_s} reads

$$\nu_{us}\frac{d}{d\nu_{us}}\alpha_{V_s} = \frac{2}{3}\frac{\alpha_s}{\pi}V_A^2\left(\left(\frac{C_A}{2} - C_f\right)\alpha_{V_o} + C_f\alpha_{V_s}\right)^3 + O(\tilde{V}_{-1}^4\tilde{V}_0, \tilde{V}_0^2\tilde{V}_{-1}^3)\,. \quad (5)$$

At higher orders in $1/m$ the same considerations than for the static limit apply here as far as the non-triangularity of the RG equations is concerned. In general, we have the structure ($\tilde{V}_m^{(0)} \equiv \tilde{V}_m$)

$$\nu_{us}\frac{d}{d\nu_{us}}\tilde{V}_m^{(n)} \sim \sum_{\{n_i\}\{m_i\}} \tilde{V}_{m_1}^{(n_1)}\tilde{V}_{m_2}^{(n_2)}\cdots\tilde{V}_{m_j}^{(n_j)}\,, \quad \text{with} \quad \sum_{i=1}^{j} n_i = n\,,\ \sum_{i=1}^{j} m_i = m\,, \quad (6)$$

and one has to pick up the leading contributions from all the possible terms. Actually, as far as the NNLL heavy quarkonium mass is concerned, the relevant ultrasoft running can be obtained by computing the diagram displayed in Fig. 1 (see [8]) (one also has to consider the running of V_A, which happens to be zero).

At the end of the day, we would have $\tilde{V}(c(1/r), d(\nu_p, 1/r), 1/r, \nu_{us}, r)$, where the running on ν_{us} is known and also the running in $1/r$ if the d is ν_p-independent. So far, the only explicit dependence of the potential on ν_p appears in the d. Nevertheless, the potential is also implicitly dependent on the three-momentum of the heavy quarks through the requirement $1/r \sim \mathbf{p} \ll \nu_p$, and also through ν_{us}, since ν_{us} needs to fulfill $\mathbf{p}^2/m \ll \nu_{us} \ll |\mathbf{p}|$ in order to ensure that only soft degrees of freedom have been integrated out for a given $|\mathbf{p}|$. This latter requirement holds if we fix $\nu_{us} = \nu_p^2/m$ (this constraint tells you how much you can run down ν_{us} in the potential before finding the cutoff ν_p^2/m caused by the cutoff of $\mathbf{p}$).

Within pNRQCD, the potentials should be introduced in the Schrödinger equation. This means that integrals over the relative three-momentum of the heavy quarks take place. When these integrals are finite one has $\mathbf{p} \sim 1/r \sim m\alpha_s$ and $\mathbf{p}^2/m \sim m\alpha_s^2$. Therefore, one can lower ν_{us} down to $\sim m\alpha_s^2$ reproducing the results obtained in Ref. [2]. In some cases, in particular in heavy quarkonium creation, the integrals over $\mathbf{p}$ are divergent, and the log structure is dictated by the ultraviolet behavior of $\mathbf{p}$ and $1/r$. This means that we can not replace $1/r$ and ν_{us} by their physical expectation values but rather by their cutoffs within the integral over $\mathbf{p}$. Therefore, for the RG equation of ν_p, the anomalous dimensions will depend (at leading order) on $\tilde{V}(c(\nu_p), d(\nu_p,\nu_p), \nu_p, \nu_p^2/m, \nu_p)$[3] and the running will go from $\nu_p \sim m$ down to $\nu_p \sim m\alpha_s$. Note that, at this stage, a single cutoff, ν_p, exists and the correlation of cutoffs can be seen. The importance of the idea that the cutoffs of the non-relativistic effective theory should be correlated was first realized in Ref. [9].

With the above discussion in mind, the matching between NRQCD and pNRQCD could be thought as follows. One does the matching by computing the potentials order by order in α_s at the matching scale $\nu_p = \nu_s = \nu_{us}$ following the procedure of Refs. [1, 6] (by doing the matching at a generic ν_p some of the running is trivially obtained). The structure of the potential at this stage then reads $\tilde{V}(c(\nu_p), d(\nu_p,\nu_p), \nu_p, \nu_p, \nu_p)$ (and similarly for the derivatives with respect $\ln r$ of the potential). This provides the starting point of the renormalization group evolution of ν_{us} (up to a constant of order one). The running of ν_{us} can then be obtained following the procedure described in Refs. [7, 2]. For the final point of the evolution of ν_{us}, we choose $\nu_{us} = \nu_p^2/m$. At the end of the day, we obtain $\tilde{V}(c(\nu_p), d(\nu_p,\nu_p), \nu_p, \nu_p^2/m, \nu_p) \equiv \tilde{V}(\nu_p)$.

The running of ν_p goes from $\nu_p = m$ (this was fixed when the matching

[3] Roughly speaking, this result can be thought as expanding $\ln r$ around $\ln \nu_p$ in the potential ie.

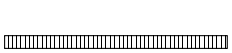
$$\begin{aligned}\tilde{V}(c(1/r), d(\nu_p, 1/r), 1/r, \nu_p^2/m, r) \;&\simeq\; \tilde{V}(c(\nu_p), d(\nu_p,\nu_p), \nu_p, \nu_p^2/m, \nu_p) \\ &\quad + \ln(\nu_p r) r \frac{d}{dr}\tilde{V}\Big|_{1/r=\nu_p} + \cdots . \end{aligned} \tag{7}$$

The $\ln(\nu_p r)$ terms give subleading contributions to the anomalous dimension when introduced in divergent integrals over $\mathbf{p}$. A more precise discussion would require a full detailed study within dimensional regularization. An explicit example of this type of corrections appear in the computation of the hyperfine splitting of the heavy quarkonium at NLL [10, 11].

between QCD and NRQCD was done) up to the physical scale of the problem $\nu_p \sim m\alpha_s$. If the running of the NRQCD matching coefficients is known, the above result gives the complete running of the potentials. The procedure to get the running of the c is known at any finite order. For the d it is just known at one-loop order, since, at this order, it is only ν_s-dependent. Nevertheless, at higher orders, dependence on ν_p appears. Therefore, the above method is not complete unless an equation for the running of ν_p is provided. This is naturally given within pNRQCD. It appears through the iteration of potentials. Let us consider this situation more in detail. The propagator of the singlet is (formally)

$$\frac{1}{E - h_s}\,. \tag{8}$$

At leading order (within an strict expansion in α_s) the propagator of the singlet reads

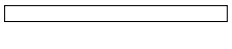


$$= G_c(E) = \frac{1}{E - h_s^{(0)}} = \frac{1}{E - \mathbf{p}^2/m - C_f\alpha_s/r}\,.$$

If we were interested in computing the spectrum at $O(m\alpha_s^6)$, one should consider the iteration of subleading potentials (δh_s) in the propagator as follows:

$$G_c(E)\delta h_s G_c(E)\cdots\delta h_s G_c(E)\,. \tag{9}$$

In general, if these potentials are singular enough, these contributions will produce logarithmic divergences due to potential loops. These divergences can be absorbed in the matching coefficients, $D^{(2)}_{d,s}$ and $D^{(2)}_{S^2,s}$, of the local potentials (those proportional to the $\delta^{(3)}(\mathbf{r})$) providing with the renormalization group equations of these matching coefficients in terms of ν_p. Let us explain how it works in detail. Since the singular behavior of the potential loops appears for $|\mathbf{p}| \gg \alpha_s/r$, a perturbative expansion in α_s is allowed in $G_c(E)$, which can be approximated by



$$= G_c^{(0)}(E) = \frac{1}{E - \mathbf{p}^2/m}\,.$$

Therefore, a practical simplification follows from the fact that the Coulomb potential, $-C_f\frac{\alpha_s}{r}$, can be considered to be a perturbation as far as the computation of the $\ln\nu_p$ ultraviolet divergences is concerned. This means that

the computation of the anomalous dimension can be organized within an expansion in α_s and using the free propagators $G_c^{(0)}$. Moreover, each $G_c^{(0)}$ produces a potential loop and one extra power of m in the numerator, which kills the powers in $1/m$ of the different potentials. This allows the mixing of potentials with different powers in $1/m$. One typical example would be the diagram in Fig. 2. The computation of this diagram would go as follows:

$$G_c^{(0)}(E)\frac{\pi C_f D_{d,s}^{(2)}}{m^2}\delta^{(3)}(\mathbf{r})G_c^{(0)}(E)C_f\frac{\alpha_{V_s}}{r}G_c^{(0)}(E)\frac{\pi C_f D_{d,s}^{(2)}}{m^2}\delta^{(3)}(\mathbf{r})G_c^{(0)}(E)\,. \quad (10)$$

Using $\delta^{(3)}(\mathbf{r}) = |\mathbf{r}=0\rangle\langle\mathbf{r}=0|$, we can see that the relevant computation reads (instead of α_{V_s} one could use α_s since the non-trivial running of α_{V_s} is a subleading effect. Nevertheless, we keep α_{V_s} since it allows to keep track of the contributions due to the Coulomb potentials)

$$\begin{aligned}&\langle\mathbf{r}=0|G_c^{(0)}(E)C_f\frac{\alpha_{V_s}}{r}G_c^{(0)}(E)|\mathbf{r}=0\rangle \quad (11)\\ &\quad\sim\int\frac{\mathrm{d}^dp'}{(2\pi)^d}\int\frac{\mathrm{d}^dp}{(2\pi)^d}\frac{m}{\mathbf{p}'^2-mE}C_f\frac{4\pi\alpha_{V_s}}{\mathbf{q}^2}\frac{m}{\mathbf{p}^2-mE}\sim -C_f\frac{m^2\alpha_{V_s}}{16\pi}\frac{1}{\epsilon},\end{aligned}$$

where $D=4+2\epsilon$ and $\mathbf{q}=\mathbf{p}-\mathbf{p}'$. This divergence is absorbed in $D_{d,s}^{(2)}$ contributing to its running at next-to-leading-log (NLL) order as follows

$$\nu_p\frac{d}{d\nu_p}D_{d,s}^{(2)}(\nu_p)\sim\alpha_{V_s}(\nu_p)D_{d,s}^{(2)2}(\nu_p)+\cdots\,. \quad (12)$$

Therefore, even without knowing the running of the d (which need to be known at NLL order in this case), we can obtain the running of the potential (one can also think of trading Eq. (12) into an equation for d, which is the only unknown parameter within the potential). This is so because $D_{d,s}^{(2)}$ is only needed with LL accuracy in the right-hand side of Eq. (12).

This finishes the procedure to the RG equations. The above method deals with the resummation of logs due to the hard, soft and ultrasoft scales. Nevertheless, for some specific kinematical situations even smaller scales could appear. Their study, however, has not yet been carried out. In any case, pNRQCD can be considered to be the right starting point to study these kinematical situations.

This line of investigation has lead to several new results on heavy quarkonium physics. They can be summarized in the following way

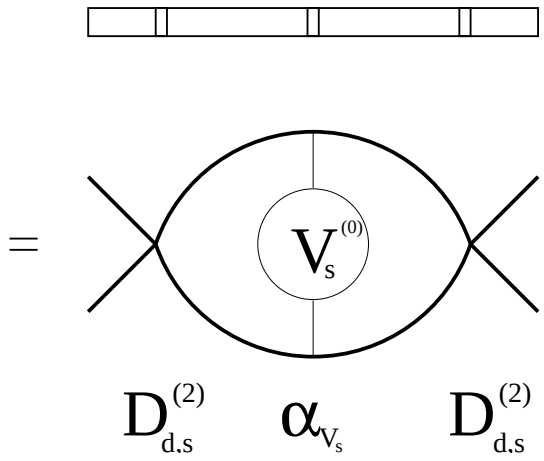


Figure 2: *One possible contribution to the running of $D^{(2)}_{d,s}$ at NLL. The first picture represents the calculation in terms of the free quark-antiquark propagator $G^{(0)}_c$ and the potentials (the small rectangles). The picture below is the representation within a more standard diagrammatic interpretation in terms of quarks and antiquarks. The delta potentials are displayed as local interactions and the Coulomb potential as an extended in space (but not in time) object.*

- The correction to the heavy quarkonium energy at NNLL [2], i.e. corrections of order

$$\delta E \sim m\alpha^4 + m\alpha^5 \ln\alpha + m\alpha^6 \ln^2\alpha + \cdots \tag{13}$$

- Corrections to the heavy quarkonium hyperfine splitting at LL [2] (first obtained in Ref. [12]) and NLL [10, 11]

$$\begin{aligned} \delta E_{HF} &\sim m\alpha^4 + m\alpha^5 \ln\alpha + m\alpha^6 \ln^2\alpha + \cdots \\ &+ m\alpha^5 + m\alpha^6 \ln\alpha + m\alpha^7 \ln^2\alpha + m\alpha^8 \ln^3\alpha + \cdots \end{aligned} \tag{14}$$

- The decays are known with NLL accuracy (this result can be easily applied to $\bar{t}-t$ production threshold or non-relativistic sum rules since the running of the electromagnetic current matching coefficient is the only non-trivial object that appears at NLL running) [13]

$$\begin{aligned} \Gamma(V_Q(nS) \to e^+e^-) &\sim m\alpha^3(1 + \alpha^2 \ln\alpha + \alpha^3 \ln^2\alpha + \cdots) \\ \Gamma(P_Q(nS) \to \gamma\gamma) &\sim m\alpha^3(1 + \alpha^2 \ln\alpha + \alpha^3 \ln^2\alpha + \cdots) \end{aligned} \tag{15}$$

and for the ratio with NNLL accuracy [14]

$$\frac{\Gamma(V_Q(nS)\to e^+e^-)}{\Gamma(P_Q(nS)\to \gamma\gamma)} \sim 1+\alpha^2\ln\alpha+\alpha^3\ln^2\alpha+\cdots \qquad (16)$$
$$+\alpha^3\ln\alpha+\alpha^4\ln^2\alpha+\cdots$$

There has also been a lot of work on the resummation of logarithms using the vNRQCD framework. Unfortunately, its first formulation suffered from some mistakes (in particular concerning the treatment of ultrasoft modes), which lead to incorrect results for the heavy quarkonium mass at NNLL [12] and the electromagnetic current matching coefficient at NLL [15]. Fortunately, they have been solved in Ref. [16] and their results now agree with those obtained in pNRQCD [2, 13].

These results may have an important phenomenological impact in several situations. Let us enumerate a few of them. The determination of the bottom and charm masses (using the experimental value of the ground state heavy quarkonium masses or non-relativistic sum rules). The determination of the $\eta_b(1S)$ mass, the hyperfine splitting of the ground state B_c system, or theoretical improved determinations of the η_c. One can also try to obtain improved determinations for the inclusive electromagnetic decays of the heavy quarkonium. On the other hand the application of this program to t-$\bar{t}$ production near threshold at the Next Linear Collider is one of the main motivations to undergo these computations. Moreover, it would also be interesting to try to perform the resummation of logarithms in semi-inclusive radiative decays [17]. A review on the phenomenological consequences of these results is presented elsewhere.

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